ARKIV FÖR FYSIK Band 16 nr 40

Communicated 14 October 1959 by Oskar Klein

Construction of wave equations by Eriksson's spinor formalism

By Bengt Enflo

1. Introduction

When setting up a relativistic wave equation for a spin-$\frac{1}{2}$-particle, Dirac [1] postulates that the equation should be of first order with constant coefficients and that its solutions should also be solutions to the Klein-Gordon equation. These postulates automatically lead to the well-known Dirac equation. After having established the equation, Dirac investigates what transformation properties the wave function should have for the equation to be Lorentz-covariant. By a somewhat different procedure, Weyl [2] is lead to the same equation. By group-theoretical considerations he is able to establish four bilinear forms by two four-spinors. These forms transform like the four space-time coordinates under proper and improper Lorentz transformations, which do not inverse the direction of time. Weyl is then able to construct a Lorentz-invariant differential operator which is identical with that found by Dirac. Eriksson's [3] treatment of these things is similar to that of Weyl in the respect that he starts with the spinors and their transformation properties. By the spinor components he constructs bilinear forms, which transform like scalars and vector components under Lorentz transformations. In order to get a firm basis for the treatment of space reflection and time reversal, Eriksson builds up the spinor representation of rotations of a six-dimensional space with four space-like and two time-like coordinates. Since every transformation belonging to the unrestricted Lorentz group is a special case of a continuous six-dimensional rotation, it is possible to deduce the behaviour of spinors under unrestricted Lorentz transformations. As to the treatment of the interaction between electrons and the electro-magnetic field, Dirac and Weyl, following the classical analogy, replace $\partial/\partial x^{\mu}$ with $\partial/\partial x^{\mu} - i\,e\,U_{\mu}$, where U_{μ} is the electro-magnetic four-potential. Eriksson ([3], p. 354) starts with a consideration of those possibilities which the established bilinear forms give him to construct a Lagrangian for a particle interacting with an electro-magnetic field. Eriksson's procedure leads, after some transformations, to the ordinary Dirac equation.

2. The spinor formalism. Vectors and scalars

In an earlier work [4] Eriksson develops the spinor representation of rotations of a five-dimensional space with four space-like dimensions and one time-like. The connexion between the coefficients of the rotation and the spinor transformation is obtained by a generalization in five dimensions of the well-known method of representing three-dimensional rotations by Cayley-Klein parameters (see e.g. Goldstein, Classical Mechanics, p. 109). In three dimensions we have four complex parameters, restricted by one complex and three real conditions, so that three real parameters are still independent. Eriksson ([4], p. 4) gets eight complex parameters, restricted by three complex conditions, so that ten independent real parameters remain. As is well-known, an n-dimensional rotation is determined by $\frac{1}{2}n(n-1)$ parameters. Those eight complex parameters are coefficients in a four-spinor transformation. Eriksson's analogous calculations for six dimensions are still unpublished, but some results are given in [3]. By means of two four-spinors, u and v (as to the transformation of a four-spinor see [3], p. 352, which is a generalization in six dimensions of [4], p. 6), Eriksson establishes the following bilinear forms on which the present investigation will be based.

1) The contravariant four-vector:

$$
\begin{aligned}
A^1 &= u_1 v_3 + u_2 v_4 - j^2 u_3 v_1 - j^2 u_4 v_2 \\
A^2 &= i(u_1 v_3 - u_2 v_4 - j^2 u_3 v_1 + j^2 u_4 v_2) \\
A^3 &= u_1 v_4 - u_2 v_3 + j^2 u_3 v_2 - j^2 u_4 v_1 \\
A^4 &= u_1 v_4 + u_2 v_3 - j^2 u_3 v_2 - j^2 u_4 v_1
\end{aligned}
$$

The parameter j is introduced by Eriksson as a factor in the "space reflection part" of the spinor representation of a five-dimensional rotation (cf. [3], p. 350 and [4], p. 6). The introduction of j is made in order to investigate the number of representations of space reflection. It turns out ([3], p. 352) that $j^4 = 1$. If $j^2 = 1$, the identical transformation and space reflection can be continuously transferred into each other by means of rotations of five-dimensional space, but if $j^2 = -1$, they cannot.

2) The quasi-vector:

$$
\begin{aligned}
C^1 &= \bar{u}_1 v_2 + \bar{u}_2 v_1 + \bar{u}_3 v_4 + \bar{u}_4 v_3 \\
C^2 &= -i(\bar{u}_1 v_2 - \bar{u}_2 v_1 + \bar{u}_3 v_4 - \bar{u}_4 v_3) \\
C^3 &= \bar{u}_1 v_1 - \bar{u}_2 v_2 - \bar{u}_3 v_3 + \bar{u}_4 v_4 \\
C^4 &= \bar{u}_1 v_1 + \bar{u}_2 v_2 + \bar{u}_3 v_3 + \bar{u}_4 v_4
\end{aligned}
$$

We will use the expression quasi-vector for a vector whose transformation coefficients are $\Delta_5 a^\mu{}_\nu$, where $a^\mu{}_\nu$ are the transformation coefficients of an ordinary vector. Δ_5 is the determinant of the Lorentz transformation in five-dimensional space. The two types of time reversal can be written

$$
\begin{aligned}
\text{a)}\quad & x_k' = x_k \, (k = 1, 2, 3, 5) \\
& x_4' = -x_4 \\
& x_6' = -x_6 \quad (\Delta_6 \text{ must be} = 1) \\
\text{b)}\quad & x_k' = -x_k \, (k = 1, 2, 3, 5) \\
& x_4' = -x_4 \\
& x_6' = -x_6
\end{aligned}
$$

Thus the negative sign is valid if we make a Lorentz transformation which can be continuously transferred into time reversal or time reversal + space reflection. In other cases $\Delta_5 = +1$. (A more detailed account of similar questions is given in [3], p. 349–354.)

3) The quasi-scalar:

$$n = u_1 v_2 - u_2 v_1 + j^2 u_3 v_4 - j^2 u_4 v_3$$

4) The scalar:

$$m = \bar{u}_1 v_3 - \bar{u}_2 v_4 - \bar{u}_3 v_1 + \bar{u}_4 v_2$$

It might have interest to investigate, whether this generalized spinor formalism can give an electron theory which is different from that which is deduced from e.g. Weyl's spinor formalism. In this paper, we will investigate all the possibilities that the above-mentioned vectors and scalars give us for constructing a Lagrangian of a spin-$\frac{1}{2}$-particle interacting with an electro-magnetic field.

When constructing a Lagrange function, Eriksson ([3], p. 354) excludes the case $j^2 = -1$. He puts the derivation symbol $\partial_\mu = \partial / \partial x^\mu$ between the two factors in each term of A^μ and uses C in that part of the Lagrangian which contains the electromagnetic interaction. Then v is put equal to u. If one assumes that the electromagnetic four-potential behaves like a quasi-vector, Eriksson's Lagrangian will be a true invariant and this is the only true invariant possible to construct. As is mentioned before, it leads to the ordinary Dirac equations. If we, however, do not require that the theory should be invariant for transformations of the type $x'_k = \pm x_k \, (k = 1, 2, 3) \, t' = -t$, we have other possibilities for constructing a Lagrangian.

3. The case $j^2 = +1$

If we put $j^2 = 1$ and $u = v$, the vector A vanishes identically. Thus we can use only C in the electro-magnetic interaction terms. It is, however, possible to use *both* A and C in the terms containing the derivatives. Such a theory would, of course, not be invariant for time reversal. Because the quasi-scalar n vanishes independently of j^2 if $u = v$, the scalar m must be used in the mass term. Since n vanishes, it is impossible, for finite mass, to construct a quasi-invariant Lagrangian, whether A is considered as a quasi-vector or not. The Lagrangian which we begin with is then:

$$\begin{aligned} L = i \{ & \bar{u}_1 [(\partial_1 - i \partial_2) u_2 + (\partial_3 + \partial_4) u_1] + \bar{u}_2 [(\partial_1 + i \partial_2) u_1 - (\partial_3 - \partial_4) u_2] + \\ & + \bar{u}_3 [(\partial_1 - i \partial_2) u_4 - (\partial_3 - \partial_4) u_3] + \bar{u}_4 [(\partial_1 + i \partial_2) u_3 + (\partial_3 + \partial_4) u_4] \} + \\ & + \varepsilon \{ u_1 [(\partial_1 + i \partial_2) u_3 + (\partial_3 + \partial_4) u_4] + u_2 [(\partial_1 - i \partial_2) u_4 - (\partial_3 - \partial_4) u_3] \\ & + u_3 [-(\partial_1 + i \partial_2) u_1 + (\partial_3 - \partial_4) u_2] + u_4 [-(\partial_1 - i \partial_2) u_2 - (\partial_3 + \partial_4) u_1] \} + \text{compl. conj.} \\ & + 2 i \varkappa \cdot (\bar{u}_1 u_3 - \bar{u}_2 u_4 - \bar{u}_3 u_1 + \bar{u}_4 u_2) \\ & + 2 U_1 \beta (\bar{u}_1 u_2 + \bar{u}_2 u_1 + \bar{u}_3 u_4 + \bar{u}_4 u_3) - 2 i U_2 \beta (\bar{u}_1 u_2 - \bar{u}_2 u_1 + \bar{u}_3 u_4 - \bar{u}_4 u_3) \\ & + 2 U_3 \beta (\bar{u}_1 u_1 - \bar{u}_2 u_2 - \bar{u}_3 u_3 + \bar{u}_4 u_4) + 2 U_4 \beta (\bar{u}_1 u_1 + \bar{u}_2 u_2 + \bar{u}_3 u_3 + \bar{u}_4 u_4) \end{aligned} \tag{1}$$

U_μ is the electro-magnetic four-potential and ε is a numerical constant which can be complex. $\varkappa$ is proportional to a mass and β to a charge. From this Lagrangian we cannot obtain Eriksson's case as a special case because the coefficient of that part of the Lagrangian which contains the derivation symbol in C is 1.

From the field equations

$$\frac{\partial L}{\partial u_r} - \partial_\mu \frac{\partial L}{\partial(\partial u_r/\partial x^\mu)} = 0$$

$$\frac{\partial L}{\partial \bar{u}_r} - \partial_\mu \frac{\partial L}{\partial(\partial \bar{u}_r/\partial x^\mu)} = 0$$

the following equations and their conjugates are derived:

$$\begin{aligned}
&(\partial_1 + i\partial_2) u_1 - (\partial_3 - \partial_4) u_2 - i\beta (U_1 + iU_2) u_1 + i\beta (U_3 - U_4) u_2 - i\bar{\varepsilon}\{(\partial_1 + i\partial_2)\bar{u}_4 - \\
&\qquad - (\partial_3 - \partial_4)\bar{u}_3\} - \varkappa u_4 = 0 \\
&(\partial_1 - i\partial_2) u_2 + (\partial_3 + \partial_4) u_1 - i\beta (U_1 - iU_2) u_2 - i\beta (U_3 + U_4) u_1 - i\bar{\varepsilon}\{(\partial_1 - i\partial_2)\bar{u}_3 + \\
&\qquad + (\partial_3 + \partial_4)\bar{u}_4\} + \varkappa u_3 = 0 \\
&(\partial_1 + i\partial_2) u_3 + (\partial_3 + \partial_4) u_4 - i\beta (U_1 + iU_2) u_3 - i\beta (U_3 + U_4) u_4 + i\bar{\varepsilon}\{(\partial_1 + i\partial_2)\bar{u}_2 + \\
&\qquad + (\partial_3 + \partial_4)\bar{u}_1\} + \varkappa u_2 = 0 \\
&(\partial_1 - i\partial_2) u_4 - (\partial_3 - \partial_4) u_3 - i\beta (U_1 - iU_2) u_4 + i\beta (U_3 - U_4) u_3 + i\bar{\varepsilon}\{(\partial_1 - i\partial_2)\bar{u}_1 - \\
&\qquad - (\partial_3 - \partial_4)\bar{u}_2\} - \varkappa u_1 = 0
\end{aligned} \tag{2}$$

We get the equations obtained by Eriksson ([3], p. 354) by putting $\beta = e$,[1] $i\bar{\varepsilon} = 1$ and not including the two first terms (which come from that part of the Lagrangian which contains the derivation symbols in C) in each equation. The current-density vector S is the above-mentioned vector C multiplied by 2β for the case $v = u$. The canonical energy-momentum tensor is:

$$\begin{aligned}
T_\mu^{\ 1} &= i[\bar{u}_1 D'_\mu u_2 + \bar{u}_2 D'_\mu u_1 + \bar{u}_3 D'_\mu u_4 + \bar{u}_4 D'_\mu u_3 - i\varepsilon\{u_1 \partial_\mu u_3 + u_2 \partial_\mu u_4 - u_3 \partial_\mu u_1 - \\
&\qquad - u_4 \partial_\mu u_2\}] + \text{compl. conj.} \\
T_\mu^{\ 2} &= i[-i\{\bar{u}_1 D'_\mu u_2 - \bar{u}_2 D'_\mu u_1 + \bar{u}_3 D'_\mu u_4 - \bar{u}_4 D'_\mu u_3\} + \varepsilon\{u_1 \partial_\mu u_3 - u_2 \partial_\mu u_4 - \\
&\qquad - u_3 \partial_\mu u_1 + u_4 \partial_\mu u_2\}] + \text{compl. conj.} \\
T_\mu^{\ 3} &= i[\bar{u}_1 D'_\mu u_1 - \bar{u}_2 D'_\mu u_2 - \bar{u}_3 D'_\mu u_3 + \bar{u}_4 D'_\mu u_4 - i\varepsilon\{u_1 \partial_\mu u_4 - u_2 \partial_\mu u_3 + u_3 \partial_\mu u_2 - \\
&\qquad - u_4 \partial_\mu u_1\}] + \text{compl. conj.} \\
T_\mu^{\ 4} &= i[\bar{u}_1 D'_\mu u_1 + \bar{u}_2 D'_\mu u_2 + \bar{u}_3 D'_\mu u_3 + \bar{u}_4 D'_\mu u_4 - i\varepsilon\{u_1 \partial_\mu u_4 + u_2 \partial_\mu u_3 - u_3 \partial_\mu u_2 - \\
&\qquad - u_4 \partial_\mu u_1\}] + \text{compl. conj.}
\end{aligned} \tag{3}$$

Here we have used the notation $D'_\mu = \partial_\mu - i\beta U_\mu$. With the aid of the wave equations and their conjugates, the following relations are established:

$$\partial_\nu S^\nu = 0 \tag{4}$$

$$\partial_\nu T_\mu^{\ \nu} = (\partial_\nu U_\mu - \partial_\mu U_\nu) S^\nu = -F_{\mu\nu} S^\nu \tag{5}$$

---

[1] e is the particle charge together with its sign. Atomic units are used.

i.e. the charge and the energy are conserved. The Lagrangian (1) was constructed without paying any regard to the gauge invariance of the theory. It can be shown (see e.g. [5], p. 76) that if gauge invariance is postulated, then (4) follows. The question whether the present theory can be formulated in a gauge-invariant form is not satisfactorily answered by (4). In order to investigate the gauge invariance and compare the present theory with the ordinary Dirac theory, the following transformation has proved to be suitable (cf. [3], p. 354).

$$\left\{\begin{aligned} \varphi_1 &= u_1 + k\,\bar{u}_4 \\ \varphi_2 &= u_2 + k\,\bar{u}_3 \\ \varphi_3 &= u_3 - k\,\bar{u}_2 \\ \varphi_4 &= u_4 - k\,\bar{u}_1 \end{aligned}\right. \tag{6}$$

where
$$k = \frac{i\,(1 - \sqrt{1 + |\varepsilon|^2})}{\varepsilon}$$

This is a natural substitution because $\begin{pmatrix} \bar{u}_4 \\ \bar{u}_3 \\ -j^2\,\bar{u}_2 \\ -j^2\,\bar{u}_1 \end{pmatrix}$ is transformed like $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$

if time reversal is disregarded.

Hence

$$\sqrt{1+|\varepsilon|^2}\,(\partial_1 + i\,\partial_2)\,\varphi_1 - \sqrt{1+|\varepsilon|^2}\,(\partial_3 - \partial_4)\,\varphi_2 - i\,\beta\,(U_1 + i\,U_2)\,\varphi_1 + i\,\beta\,(U_3 - U_4)\,\varphi_2 - \varkappa\,\varphi_4 = 0$$

$$\sqrt{1+|\varepsilon|^2}\,(\partial_1 - i\,\partial_2)\,\varphi_2 + \sqrt{1+|\varepsilon|^2}\,(\partial_3 + \partial_4)\,\varphi_1 - i\,\beta\,(U_1 - i\,U_2)\,\varphi_2 - i\,\beta\,(U_3 + U_4)\,\varphi_1 + \varkappa\,\varphi_3 = 0$$

$$\sqrt{1+|\varepsilon|^2}\,(\partial_1 + i\,\partial_2)\,\varphi_2 + \sqrt{1+|\varepsilon|^2}\,(\partial_3 + \partial_4)\,\varphi_4 - i\,\beta\,(U_1 + i\,U_2)\,\varphi_3 - i\,\beta\,(U_3 + U_4)\,\varphi_4 + \varkappa\,\varphi_2 = 0$$

$$\sqrt{1+|\varepsilon|^2}\,(\partial_1 - i\,\partial_2)\,\varphi_4 - \sqrt{1+|\varepsilon|^2}\,(\partial_3 - \partial_4)\,\varphi_3 - i\,\beta\,(U_1 - i\,U_2)\,\varphi_4 + i\,\beta\,(U_3 - U_4)\,\varphi_3 - \varkappa\,\varphi_1 = 0 \tag{7}$$

From these equations we get the ordinary Dirac equations (cf. [3], p. 355) putting $\beta = e\sqrt{1+|\varepsilon|^2}$ and $\varkappa = m\sqrt{1+|\varepsilon|^2}$ where e is the particle charge and m is the particle mass.

$$\begin{aligned} (D_1 + i\,D_2)\,\varphi_1 - (D_3 - D_4)\,\varphi_2 - m\,\varphi_4 &= 0 \\ (D_1 - i\,D_2)\,\varphi_2 + (D_3 + D_4)\,\varphi_1 + m\,\varphi_3 &= 0 \\ (D_1 + i\,D_2)\,\varphi_3 + (D_3 + D_4)\,\varphi_4 + m\,\varphi_2 &= 0 \\ (D_1 - i\,D_2)\,\varphi_4 - (D_3 - D_4)\,\varphi_3 - m\,\varphi_1 &= 0 \end{aligned} \tag{8}$$

We use the notation $D_\mu = \partial_\mu - i\,e\,U_\mu$.

The transformation

$$\begin{cases} \psi_1 = \varphi_1 - i\varphi_3 \\ \psi_2 = \varphi_2 + i\varphi_4 \\ \psi_3 = \varphi_1 + i\varphi_3 \\ \psi_4 = \varphi_2 - i\varphi_4 \end{cases}$$

leads to the more well-known form

$$(D_0 + \boldsymbol{\alpha}\cdot\boldsymbol{D} + i\,m\,\alpha_0)\,\psi = 0$$

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad k = 1, 2, 3$$

$$\alpha_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli spin matrices.

It is very easy to see that the above-mentioned choice of the constants $\beta, \varkappa$ is the only possible. In order to show this we write out, in Minkowski notation, the second order equation that follows from eqs. (7):

$$\left[\sum_\mu (\partial_\mu - i\frac{\beta}{\sqrt{1+|\varepsilon|^2}}U_\mu)^2 - \frac{\varkappa^2}{1+|\varepsilon|^2} + i\frac{\beta}{2\sqrt{1+|\varepsilon|^2}}\sum_{\mu,\nu}\gamma_\mu\gamma_\nu F_{\mu\nu}\right]\varphi = 0 \tag{9}$$

In absence of an electro-magnetic field this equation must be the Klein-Gordon equation. Therefore we must have $\frac{\varkappa^2}{1+|\varepsilon|^2} = m^2$. In order to get the correct magnetic moment $\frac{e}{2m}$ of the electron we must require the relation $\frac{\beta}{\sqrt{1+|\varepsilon|^2}} = e$. The other possibility is to assume another value than 1 for $\hbar$, but this choice does not seem very attractive from a physical point of view and has nothing to do with the spinor formalism used in this paper because of the possibility of transforming eqs. (2) into eqs. (7).

4. The case $j^2 = -1$

If $j^2 = -1$ the insertion of derivatives in A gives a four-divergence. It can be shown (see e.g. [6], p. 94) that the addition of a four-divergence to the Lagrangian does not alter the equations of motion. Thus only C can be used in that part of the Lagrangian which contains the derivatives. Instead, it is now possible to use both A and C in the electro-magnetic interaction part of the Lagrangian. Thus we start with the following Lagrangian:

$$\begin{aligned} L = i\{&\bar{u}_1[(\partial_1 - i\partial_2)u_2 + (\partial_3 + \partial_4)u_1] + \bar{u}_2[(\partial_1 + i\partial_2)u_1 - (\partial_3 - \partial_4)u_2] + \\ &+ \bar{u}_3[(\partial_1 - i\partial_2)u_4 - (\partial_3 - \partial_4)u_3] + \bar{u}_4[(\partial_1 + i\partial_2)u_3 + (\partial_3 + \partial_4)u_4]\} + \text{compl. conj.} \\ &+ 2i\varkappa(\bar{u}_1 u_3 - \bar{u}_2 u_4 - \bar{u}_3 u_1 + \bar{u}_4 u_2) + 2f[U_1(\bar{u}_1 u_2 + \bar{u}_2 u_1 + \bar{u}_3 u_4 + \bar{u}_4 u_3) - \\ &- iU_2(\bar{u}_1 u_2 - \bar{u}_2 u_1 + \bar{u}_3 u_4 - \bar{u}_4 u_3) + U_3(\bar{u}_1 u_1 - \bar{u}_2 u_2 - \bar{u}_3 u_3 + \bar{u}_4 u_4) + \end{aligned}$$

$$\begin{aligned}&+U_4(\bar u_1 u_1+\bar u_2 u_2+\bar u_3 u_3+\bar u_4 u_4)]+2d[U_1(u_1u_3+u_2u_4)+iU_2(u_1u_3-u_2u_4)+\\&\quad+U_3(u_1u_4-u_2u_3)+U_4(u_1u_4+u_2u_3)]+2\bar d[U_1(\bar u_1\bar u_3+\bar u_2\bar u_4)-iU_2(\bar u_1\bar u_3-\bar u_2\bar u_4)+\\&+U_3(\bar u_1\bar u_4-\bar u_2\bar u_3)+U_4(\bar u_1\bar u_4+\bar u_2\bar u_3)]\qquad(10)\end{aligned}$$

$\varkappa$ is proportional to a mass and d and f to a charge. d has the character of a charge multiplied with a complex number. The following equations are derived from the Lagrangian:

$$\begin{aligned}&(\partial_1+i\partial_2)u_1-(\partial_3-\partial_4)u_2-if[(U_1+iU_2)u_1-(U_3-U_4)u_2]-i\bar d[(U_1+iU_2)\bar u_4-\\&\qquad\qquad-(U_3-U_4)\bar u_3]-\varkappa u_4=0\\&(\partial_1-i\partial_2)u_2+(\partial_3+\partial_4)u_1-if[(U_1-iU_2)u_2+(U_3+U_4)u_1]-i\bar d[(U_1-iU_2)\bar u_3+\\&\qquad\qquad+(U_3+U_4)\bar u_4]+\varkappa u_3=0\\&(\partial_1+i\partial_2)u_3+(\partial_3+\partial_4)u_4-if[(U_1+iU_2)u_3+(U_3+U_4)u_4]-i\bar d[(U_1+iU_2)\bar u_2+\\&\qquad\qquad+(U_3+U_4)\bar u_1]+\varkappa u_2=0\\&(\partial_1-i\partial_2)u_4-(\partial_3-\partial_4)u_3-if[(U_1-iU_2)u_4-(U_3-U_4)u_3]-i\bar d[(U_1-iU_2)\bar u_1-\\&\qquad\qquad-(U_3-U_4)\bar u_2]-\varkappa u_1=0\;(11)\end{aligned}$$

In this case, the current-density vector is

$$\begin{aligned}\tfrac12 S^1&=f(\bar u_1u_2+\bar u_2u_1+\bar u_3u_4+\bar u_4u_3)+d(u_1u_3+u_2u_4)+\bar d(\bar u_1\bar u_3+\bar u_2\bar u_4)\\\tfrac12 S^2&=-if(\bar u_1u_2-\bar u_2u_1+\bar u_3u_4-\bar u_4u_3)+id(u_1u_3-u_2u_4)-i\bar d(\bar u_1\bar u_3-\bar u_2\bar u_4)\\\tfrac12 S^3&=f(\bar u_1u_1-\bar u_2u_2-\bar u_3u_3+\bar u_4u_4)+d(u_1u_4-u_2u_3)+\bar d(\bar u_1\bar u_4-\bar u_2\bar u_3)\\\tfrac12 S^4&=f(\bar u_1u_1+\bar u_2u_2+\bar u_3u_3+\bar u_4u_4)+d(u_1u_4+u_2u_3)+\bar d(\bar u_1\bar u_4+\bar u_2\bar u_3)\qquad(12)\end{aligned}$$

and the canonical energy-momentum tensor:

$$\begin{aligned}T_\mu^1&=i[\bar u_1D''_\mu u_2+\bar u_2D''_\mu u_1+\bar u_3D''_\mu u_4+\bar u_4D''_\mu u_3]+2d(u_1U_\mu u_3+u_2U_\mu u_4)+\\&\qquad+\text{compl. conj.}\\T_\mu^2&=i[-i(\bar u_1D''_\mu u_2-\bar u_2D''_\mu u_1+\bar u_3D''_\mu u_4-\bar u_4D''_\mu u_3)]+2id(u_1U_\mu u_3-u_2U_\mu u_4)+\\&\qquad+\text{compl. conj.}\\T_\mu^3&=i[\bar u_1D''_\mu u_1-\bar u_2D''_\mu u_2-\bar u_3D''_\mu u_3+\bar u_4D''_\mu u_4]+2d(u_1U_\mu u_4-u_2U_\mu u_3)+\\&\qquad+\text{compl. conj.}\\T_\mu^4&=i[\bar u_1D''_\mu u_1+\bar u_2D''_\mu u_2+\bar u_3D''_\mu u_3+\bar u_4D''_\mu u_4]+2d(u_1U_\mu u_4+u_2U_\mu u_3)+\\&\qquad+\text{compl. conj.}\qquad(13)\end{aligned}$$

where $$D''_\mu=\partial_\mu-ifU^\mu$$

In the same way as in the first case the equations (4) and (5) can be verified for the new S^ν and T_μ^ν. If $f>|d|$, the charge density S^4 is positive definite.

It has been found that eqs. (11) can also be written as Dirac equations if the following substitution is made:

$$\begin{cases} \varphi_1 = u_1 + \lambda \bar{u}_4 \\ \varphi_2 = u_2 + \lambda \bar{u}_3 \\ \varphi_3 = u_3 + \lambda \bar{u}_2 \\ \varphi_4 = u_4 + \lambda \bar{u}_1 \end{cases} \tag{14}$$

where
$$\lambda = \frac{f - \sqrt{f^2 - |d|^2}}{d}$$

Hence

$$\begin{aligned} &(\partial_1 + i\partial_2)\varphi_1 - (\partial_3 - \partial_4)\varphi_2 - i\sqrt{f^2 - |d|^2}(U_1 + iU_2)\varphi_1 + i\sqrt{f^2 - |d|^2}(U_3 - U_4)\varphi_2 - \\ &\qquad - \varkappa\varphi_4 = 0 \\ &(\partial_1 - i\partial_2)\varphi_2 + (\partial_3 + \partial_4)\varphi_1 - i\sqrt{f^2 - |d|^2}(U_1 - iU_2)\varphi_2 - i\sqrt{f^2 - |d|^2}(U_3 + U_4)\varphi_1 + \\ &\qquad + \varkappa\varphi_3 = 0 \\ &(\partial_1 + i\partial_2)\varphi_3 + (\partial_3 + \partial_4)\varphi_4 - i\sqrt{f^2 - |d|^2}(U_1 + iU_2)\varphi_3 - i\sqrt{f^2 - |d|^2}(U_3 + U_4)\varphi_4 + \\ &\qquad + \varkappa\varphi_2 = 0 \\ &(\partial_1 - i\partial_2)\varphi_4 - (\partial_3 - \partial_4)\varphi_3 - i\sqrt{f^2 - |d|^2}(U_1 - iU_2)\varphi_4 + i\sqrt{f^2 - |d|^2}(U_3 - U_4)\varphi_3 - \\ &\qquad - \varkappa\varphi_1 = 0 \end{aligned} \tag{15}$$

a) The case $f > |d|$. The ordinary Dirac equations are obtained by putting $\sqrt{f^2 - |d|^2} = e$ and $\varkappa = m$. That this is the only possibility of choosing the constants is seen in the same way as in the case $j^2 = 1$.

b) The case $f \leqslant |d|$. If $f \leqslant |d|$, the substitution (14) has no inverse because λ is then of the form $e^{i\alpha}$. Hence $\varphi_4 = e^{i\alpha}\bar{\varphi}_1$, and $\varphi_3 = e^{i\alpha}\bar{\varphi}_2$. We thus have a two-component system consisting of the first two equations of (15). Since we cannot express the current-density vector (12) in the φ:s, we have abandoned the Lagrangian formalism. It does not seem as if this two-component system, at least for the case $f < |d|$, has a physical meaning, because the U-terms have then the wrong reality. The same two-component equation with the correct reality of the U-terms has been investigated by Eriksson,[1] who did not get any fine structure formula for the hydrogen atom out of it. A modification of a new kind has, however, recently given a result [7].

The special case $f = |d|$ would, however, perhaps be of interest, as it may correspond to a neutral particle. The author has desisted from making any investigation of this question and also from making any investigation of the original four-component system (11) in the case $f \leqslant |d|$ because of the impossibility of transforming this system into equations of accustomed types.

SUMMARY

A Lagrangian for a spin-$\frac{1}{2}$-particle in an electro-magnetic field is constructed by means of Eriksson's spinor formalism in the most general way that is possible. Although the Lagrangian is not invariant for time reversal the formalism leads to equations of the same type as the

---

[1] Private communication

Dirac equation, except for a special choice of the parameters, where equations are obtained, which seem to have no physical significance.

ACKNOWLEDGEMENTS

This investigation has been suggested by Prof. A. Eriksson, Director of the Division, and the author is sincerely grateful to him for helpful guidance and instructive discussions. The author is also indebted to Prof. O. Klein, University of Stockholm, for his kind interest in the work, and to Dr. S. Hjalmars for valuable advice.

Division of Mechanics II, Royal Institute of Technology, Stockholm, August 1959.

REFERENCES

1. P. A. M. Dirac, Quantum Mechanics, Oxford 1958.
2. H. Weyl, Gruppentheorie und Quantenmechanik, Leipzig 1928.
3. H. A. S. Eriksson, Ark. f. fys., *6*, No. 33 (1953).
4. H. A. S. Eriksson, Ark. f. mat. astr. fys., *29 A*, No. 14 (1943).
5. E. M. Corson, Introduction to tensors, spinors and relativistic wave equations, Glasgow 1953.
6. S. S. Schweber, H. A. Bethe, F. de Hoffman, Mesons and fields, vol. I, Fields, New York 1956.
7. H. A. S. Eriksson, Ark. f. fys., *15*, No. 2 (1959).

Tryckt den 17 februari 1960

Uppsala 1960. Almqvist & Wiksells Boktryckeri AB